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Derivation of amplitude equations by the renormalization group method

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A proper formulation in the perturbative renormalization group method is presented to deduce amplitude equations. The formulation makes it possible not only to avoid a serious difficulty in the previous reduction to amplitude equations by eliminating all of the secular terms but also to derive consistently higher-order correction to amplitude equations. [S1063-651X(97)50910-X]

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Recently a novel method based on the perturbative renormalization group (RG) theory has been developed as an asymptotic singular perturbation technique by Chen, Goldenfeld, and Oono [1]. Their renormalization group method (RGM) removes secular or divergent terms from a perturbation series by renormalizing integral constants of lower-order solutions. Although they have impressively succeeded in application to ordinary differential equations [2], a consistent formulation of the RG method has not been established for application to partial differential equations. In fact, we have recently pointed out [7] that all of the secular solutions of a perturbation series have not been eliminated yet in the previous derivation of amplitude equations from partial differential equations [2,3] .

In this paper, we present a suitable formulation of a perturbation problem in RGM in order to avoid such a difficulty and deduce some amplitude equations and their higher-order correction consistently in the framework of RGM. A crucial part of our formulation is to scale all variables except one independent variable so that the nonscaled variable is taken as a parameter of RG. Therefore, the renormalization group equation (RGE) or an amplitude equation obtained by truncation of a perturbation series depends on the choice of scale; that is, the initial setting of a perturbation problem, of which significance is pointed out by S. Sasa [8].

I. DERIVATION FROM A WEAKLY NONLINEAR WAVE EQUATION

As the first example, we derive an amplitude equation from the following weakly nonlinear wave equation:

$$\partial_t^2 u - \partial_x^2 u + (1+\epsilon^2 u^2)u = 0, \tag{1}$$

where ϵ is a small parameter. In order to focus our attention to a slowly varying amplitude, let us introduce a complex amplitude A and a stretched variable ξ,

$$u = A\exp[i(kx-\omega t)] + \text{c.c.}, \quad \xi = \epsilon x, \tag{2}$$

where k is a wave number, $\omega=\sqrt{1+k^2}$, and c.c. denotes complex conjugate, then Eq. (1) is rewritten as

$$[LA - \epsilon(2ik\partial_\xi + \epsilon\partial_\xi^2)A + 3\epsilon^2|A|^2A]e^{i\theta} + \epsilon^2A^3e^{3i\theta} + \text{c.c.} = 0, \tag{3}$$

where $L \equiv \partial_t^2 - 2i\omega\partial_t$ and $\theta = kx-\omega t$. In this formulation, t is chosen as a parameter of RG. Substituting an expansion

$$A = A_0(\xi) + \epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \cdots, \tag{4}$$

into Eq. (3) and retaining only secular contribution, we have, up to order ϵ^3,

$$LA_1 = 2ik\partial_\xi A_0,$$

$$LA_2 = 2ik\partial_\xi A_1 + \partial_\xi^2 A_0 - 3|A_0|^2A_0,$$

$$LA_3 = 2ik\partial_\xi A_2 + \partial_\xi^2 A_1 - 6|A_0|^2A_1 - 3A_0^2\overline{A_1},$$

where $\overline{A}$ is complex conjugate of A. Noting that secular (polynomial) solutions of $LP_j = t^j$ are given by

$$P_0 = \frac{it}{2\omega}, \quad P_1 = \frac{it^2}{4\omega} + \frac{t}{4\omega^2}, \quad P_2 = \frac{it^3}{6\omega} + \frac{t^2}{4\omega^2} - \frac{it}{4\omega^3},$$

we obtain the following secular solution up to order ϵ^3:

$$A = A_0 + t\left[-\epsilon\dot{\omega}\partial_\xi A_0 + \epsilon^2\frac{i}{2}\left(\ddot{\omega}\partial_\xi^2A_0 - \frac{3}{\omega}\left|A_0\right|^2A_0\right) - \epsilon^3\frac{\dot{\omega}}{2\omega}\partial_\xi\left(\ddot{\omega}\partial_\xi^2A_0 - \frac{3}{\omega}\left|A_0\right|^2A_0\right)\right] + \frac{t^2}{2}(\epsilon^2\dot{\omega})\left[\dot{\omega}\partial_\xi A_0 - i\epsilon\partial_\xi\left(\ddot{\omega}\partial_\xi^2A_0 - \frac{3}{\omega}\left|A_0\right|^2A_0\right)\right] - \epsilon^3\frac{t^3}{6}\dot{\omega}^3\partial_\xi^3A_0, \tag{5}$$

where $\dot{\omega} = d\omega/dk$ and $\ddot{\omega} = d\dot{\omega}/dk$. All of secular terms in Eq. (5) are removed by renormalizing A_0 and the renormalized amplitude A is described by RGE,

$$\partial_t A = -\epsilon\dot{\omega}\partial_\xi A + \epsilon^2\frac{i}{2}\left(\ddot{\omega}\partial_\xi^2A - \frac{3}{\omega}\left|A\right|^2A\right) - \epsilon^3\frac{\dot{\omega}}{2\omega}\partial_\xi\left(\ddot{\omega}\partial_\xi^2A - \frac{3}{\omega}\left|A\right|^2A\right), \tag{6}$$

$$\partial_t^2 A = \epsilon^2\dot{\omega}\left[\dot{\omega}\partial_\xi A - i\epsilon\partial_\xi\left(\ddot{\omega}\partial_\xi^2A - \frac{3}{\omega}\left|A\right|^2A\right)\right], \tag{7}$$

$$\partial_t^3 A = -\epsilon^3\dot{\omega}^3\partial_\xi^3 A, \tag{8}$$

where Eq. (6) is the nonlinear Schrödinger equation with correction up to order ϵ^3, while Eqs. (7) and (8) are easily shown to be compatible with Eq. (6) up to order ϵ^3. Since the above procedure continues to arbitrary order in ϵ, we can derive systematically the nonlinear Schrödinger equation with correction up to arbitrary order.

II. DERIVATION FROM THE SWIFT-HOHENBERG EQUATION

Let us derive slowly varying amplitude equations from the Swift-Hohenberg equation [4],

$$[\partial_t+(1+\triangle)^2-\epsilon^2(1-u^2)]u=0, \tag{9}$$

where $\triangle\equiv\partial_x^2+\partial_y^2$. Introducing a complex amplitude A and stretched variables τ and η as

$$u=A\exp(ix)+\text{c.c.},\quad \tau=\epsilon^m t,\quad \eta=\epsilon^n y,$$

where m and n are positive constants, we rewrite Eq. (9) as

$$[MA+2\epsilon^{2n}(2i\partial_x+\partial_x^2)\partial_\eta^2 A+\epsilon^{4n}\partial_\eta^4 A+\epsilon^m\partial_\tau A -\epsilon^2(1-3|A|^2)A]e^{ix}+\epsilon^2A^3e^{3ix}+\text{c.c.}=0, \tag{10}$$

where $M\equiv(\partial_x^2+2i\partial_x)^2$ and x is a parameter of renormalization group. When we set $m=2$ and $n=1/2$, we obtain the secular solution of Eq. (10), through the perturbative procedure similar to the example in Sec. I, as

$$A=A_0(\tau,\eta)+x\epsilon C_0(\tau,\eta)+\frac{x^2}{2}-\frac{\epsilon^2}{4}B_0+\frac{\epsilon^3}{4}\left[\frac{1}{2}\partial_\eta^2 B_0+i(\partial_\tau +\partial_\eta^4-1+6|A_0|^2)C_0+3iA_0^2\overline{C_0}\right]-\frac{x^3}{6}\frac{\epsilon^3}{4}[i\partial_\eta^2 B_0 -(\partial_\tau+\partial_\eta^4-1+6|A_0|^2)C_0-3A_0^2\overline{C_0}], \tag{11}$$

where $\overline{C}$ is complex conjugate of C and

$$B_0=-(\partial_\tau+\partial_\eta^4-1+3|A_0|^2)A_0-4i\partial_\eta^2 C_0.$$

Renormalizing $A_0\to A$, $B_0\to B$, and $C_0\to C$ so that secular terms in Eq. (11) are removed, we get the following RGE:

$$\partial_x A=\epsilon C, \tag{12}$$

$$\partial_x^2 A=-\frac{\epsilon^2}{4}B+\frac{\epsilon^3}{4}\left[\frac{1}{2}\partial_\eta^2 B+i(\partial_\tau+\partial_\eta^4-1+6|A|^2)C +3iA^2\overline{C}\right], \tag{13}$$

$$\partial_x^3 A=-\frac{\epsilon^3}{4}[i\partial_\eta^2 B-(\partial_\tau+\partial_\eta^4-1+6|A|^2)C-3A^2\overline{C}]. \tag{14}$$

Eliminating C from Eqs. (12) and (13), we have the Newell-Whitehead-Segel equation [5] with correction up to order ϵ^3,

$$\partial_x^2 A=i\epsilon\left(1-\frac{\epsilon}{2}\partial_\eta^2\right)\partial_x\partial_\eta^2 A+\frac{\epsilon^2}{4}\left(1+i\partial_x-\frac{\epsilon}{2}\partial_\eta^2\right) \times(\partial_\tau+\partial_\eta^4-1+3|A|^2)A.$$

It is easy to see that Eq. (14) is compatible with Eqs. (12) and (13). If we set $m=2$ and $n=1$, the similar procedure yields another amplitude equation up to order ϵ^3,

$$\partial_x^2 A=\frac{\epsilon^2}{4}(1+i\partial_x)(\partial_\tau-1+3|A|^2)A+i\epsilon^2\partial_x\partial_\eta^2 A.$$

III. DERIVATION FROM A MODEL EQUATION

As a final example, we derive a slow amplitude equation from the following model equation:

$$\{\partial_t[\partial_t+(k^2+\triangle)^2-\epsilon^2(1-u^2)]-\triangle\}u+(1+\epsilon^2u^2)u=0, \tag{15}$$

The nonlinear wave equation (1) and the Swift-Hohenberg equation (9) are combined in the model equation (15), which may be one of the simplest equations describing the Hopf bifurcation in continuous media. Let us choose t as a parameter of RG and introduce a complex amplitude A and a stretched variable ξ in the same way as the first example [see Eq. (2)] and $\eta=\epsilon y$, then we can derive the following two-dimensional complex Ginzburg-Landau equation [6] with correction up to order ϵ^3 after following the similar procedure used in the above examples:

$$(\partial_t+\epsilon\dot{\omega}\partial_\xi)A=\frac{i\epsilon^2}{2}B-\frac{\dot{\omega}\epsilon^3}{2\omega}\partial_\xi\left(\ddot{\omega}\partial_\xi^2+\frac{\dot{\omega}}{k}\partial_\eta^2-3|A|^2\right)A,$$

where

$$B=\left[(\ddot{\omega}-4ik^2)\partial_\xi^2+\frac{\dot{\omega}}{k}\partial_\eta^2-i-3(1-i)|A|^2\right]A.$$

In summary, we present a proper formulation in the perturbative renormalization group method and deduce typical amplitude equations and their higher-order correction consistently. In our formulation, we can avoid the serious difficulty appearing in the previous derivation of amplitude equations by changing the initial setting of a perturbation problem.

[1] L. Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. Lett. **73**, 1311 (1994).
[2] L. Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[3] R. Graham, Phys. Rev. Lett. **76**, 2185 (1996).
[4] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[5] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[6] K. Matsuba, K. Imai, and K. Nozaki, Physica D **107**, 69 (1997).
[7] K. Matsuba and K. Nozaki, J. Phys. Soc. Jpn. (to be published).
[8] S. Sasa (private communication).